On Gauge Choice of Spherically Symmetric 3-Branes

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Gauge choice for a spherically symmetric 3-brane embedded in a D-dimensional bulk with arbitrary matter fields on and off the brane is studied. It is shown that Israel's junction conditions across the brane restrict severely the dependence of the matter fields on the spacetime coordinates. As examples, a scalar field or a Yang-Mills potential can be only either time-dependent or radial-coordinate dependent for the chosen gauge, while for a perfect fluid it must be co-moving.

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I. INTRODUCTION

A number of current unification theories such as string theory and M-theory suggest that we may live in a world that has more than three spatial dimensions. Because only three of these are presently observable, one has to explain why the others are hidden from detection. One such explanation is the so-called Kaluza-Klein (KK) compactification, according to which the size of the extra dimensions is very small (often taken to be on the order of the Planck length). As a consequence, modes that have momentum in the directions of the extra dimensions are excited at currently inaccessible energies.

Recently, the braneworld scenario [1,2] has dramatically changed this point of view and, in the process, received a great deal of attention. At present, there are a number of proposed models (See, for example, [3] and references therein.). In particular, Arkani-Hamed *et al* (ADD) [1] pointed out that the extra dimensions need not necessarily be small and may even be on the scale of millimeters. This model assumes that Standard Model fields are confined to a three (spatial) dimensional hypersurface (a 3-brane) living in a larger dimensional bulk space while the gravitational field propagates in the whole bulk. Additional fields may live only on the brane or in the whole bulk, provided that their current undetectability is consistent with experimental bounds. One of the most attractive features of this model is that it may potentially resolve the long standing hierarchy problem, namely the large difference in magnitudes between the Planck and electroweak scales.

In a different model, Randall and Sundrum (RS) [2] showed that if the self-gravity of the brane is included, gravitational effects can be localized near the brane at low energy and the 4D newtonian gravity will be reproduced on the brane even in the presence of infinitely large extra dimensions. In this model, the $4D$ Planck scale, M_{Pl}, is determined by the curvature of the extra dimensions rather than by their size, as proposed in [1].

The RS model [1] was soon generalized to include arbitrary matter fields on the brane [4]. In particular, Shiromizu, Maeda and Sasaki (SMS) considered the embedding of a 3-brane $(M, {}^{(4)}g_{AB})$ in a 5D bulk $(V, {}^{(5)}g_{AB})$, where the 3-brane metric is given by ${}^{(4)}g_{AB} = {}^{(5)}g_{AB} - n_A n_B$, and n_A is the unit normal vector to the brane. Note that we use Greek indices to run from 0 to 3, uppercase Latin indices to run from 0 to $D-1$, and lowercase Latin indices to run from 0 to $D-2$. Using the Gauss–Codacci relations, SMS wrote the 5D Einstein field equations ${}^{(5)}G_{AB} = \kappa_5^2\, {}^{(5)}T_{AB}$ in the form

$$ {}^{(4)}R_{AB} - \frac{1}{2}\, {}^{(4)}g_{AB}\, {}^{(4)}R = {}^{(4)}\mathcal{T}_{AB}, \tag{1.1} $$

$$ D_C K^C_A - D_A K = \kappa_5^2\, {}^{(5)}T_{BC}\, {}^{(4)}g^C_A\, n^B, \tag{1.2} $$

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[1] In this Letter we are mainly concerned with the so-called RS2 model, in which only one brane exists.

where these equations are understood to apply in each of the two regions, $V^+(z \geq 0)$ and $V^-(z \leq 0)$, and z denotes the coordinate of the extra dimension and $z = 0$ is the location of the brane. The quantity ${}^{(4)}\mathcal{T}_{AB}$ is given by

$$
\begin{aligned}
{}^{(4)}\mathcal{T}_{AB} \equiv & \frac{2{\kappa_5}^2}{3}\left\{{}^{(5)}T_{CD}\,{}^{(4)}g^C_A\,{}^{(4)}g^D_B + \left[{}^{(5)}T_{CD}n^C n^D - \frac{1}{4}{}^{(5)}T^C_C\right]{}^{(4)}g_{AB}\right\}\\
& + KK_{AB} - K^C_A K_{BC} - \frac{1}{2}{}^{(4)}g_{AB}\left(K^2 - K^{CD}K_{CD}\right) - \mathcal{E}_{AB},\\
\mathcal{E}_{AB} \equiv & {}^{(5)}C^E_{FCD}n_E n^C\,{}^{(4)}g^F_A\,{}^{(4)}g^D_B,
\end{aligned}
\tag{1.3}
$$

and ${}^{(5)}C^A_{BCD}$ denotes the Weyl tensor of the bulk.

The boundary conditions at $z = 0$ are simply the Israel junction conditions [5],

$$
[K_{AB}]^- = -\kappa_5^2\left(S_{AB} - \frac{1}{3}{}^{(4)}g_{AB}\,S\right),
\tag{1.4}
$$

where

$$
\begin{aligned}
[K_{AB}]^- \equiv & \lim_{z\to 0^+} K^+_{AB} - \lim_{z\to 0^-} K^-_{AB},\\
S_{AB} \equiv & {}^{(4)}T_{AB} - \lambda\,{}^{(4)}g_{AB},
\end{aligned}
\tag{1.5}
$$

with λ and ${}^{(4)}T_{AB}$ being, respectively, the cosmological constant and the energy-momentum tensor on the 3-brane. Combining the assumption of Z_2 symmetry with Eqs. (1.4) and (1.1) in the limit $z \to 0^{\pm}$, SMS obtained the effective 4D Einstein field equations on the 3-brane [4]

$$
{}^{(4)}G_{AB} = -\Lambda_4\,{}^{(4)}g_{AB} + 8\pi G_4\,{}^{(4)}T_{AB} + \kappa_5^4\,\pi_{AB} - \mathcal{E}_{AB},
\tag{1.6}
$$

where $G_4 \equiv {\kappa_5}^4\lambda/(48\pi)$. These equations include two correction terms to the original Einstein equations, namely $\mathcal{E}_{AB}$ and π_{AB}. In the weak field limit, the first term gives rise to the massive KK modes of the graviton [2]. The second term, π_{AB}, is negligible for weak fields. However, in strong field situations, such as those at the threshold of black hole formation, it is expected to play a significant role. Indeed, it would appear to be this term which is the origin of the result of [6] that, given the 4D projected Einstein equations (1.6) and the Israel matching conditions (1.4), *the exterior of a collapsing homogeneous dust cloud cannot be static*. This result was further generalized to other cases [7], and represents a significant departure from the familiar result in Einstein's theory, in which the vacuum exterior must be the static Schwarzschild solution. A number of other recent results suggest additional phenomena different from the standard predictions of general relativity [8]. In particular, it was argued that static braneworld black holes might not exist at all [9].

Braneworld scenarios have further been promoted by the possibility that they may provide the origin of dark energy [10], which is needed in order to explain why our universe is currently accelerating [11].

In this Letter, we report on some results concerning the gauge choice for matter fields confined on a spherically symmetric 3-brane. We show that the boundary conditions (1.4) serve as very strong restrictions on the possible dependence of the matter fields on the spacetime coordinates. In particular, for a particular choice of the gauge, a scalar field or a Yang-Mills field can be only either time-dependent or radial-coordinate dependent, while for a perfect fluid its radial velocity must vanish. Moreover, these conclusions would appear to be true not only for the generalized RS models in a 5D bulk, but also for 3-branes in higher dimensional spacetimes [3]. This is quite different from its four-dimensional counterpart.

Before showing these results, let us first give a brief review on the gauge choice of a 4D spacetime with spherical symmetry, for which the general metric can be cast, without loss of generality, in the form,

$$
ds_4^2 = g_{ab}\left(x^c\right)dx^a dx^b + s^2\left(x^c\right)d\Omega^2,\ (a, b, c = 0, 1),
\tag{1.7}
$$

where $d\Omega^2 \equiv d\theta^2 + \sin^2\theta d\varphi^2$. Clearly, the form of the metric is invariant under the coordinate transformations,

$$
x^0 = x^0\left({x'}^0, {x'}^1\right), \quad x^1 = x^1\left({x'}^0, {x'}^1\right).
\tag{1.8}
$$

Using these two degrees of the gauge freedom, one can choose different gauges for different matter fields. Because of the complexity of the Einstein field equations, such choices often are crucial in studying the problem. For example, for a collapsing perfect fluid, one usually chooses the so-called comoving gauge,

$$g_{01}\left(x^{c}\right)=0, \quad u_{A}=\left(-g_{00}\right)^{1/2} \delta_{A}^{0}, \tag{1.9}$$

so that the metric takes the form,

$$ds_{4}^{2}=g_{00}\left(x^{c}\right)\left(dx^{0}\right)^{2}+g_{11}\left(x^{c}\right)\left(dx^{1}\right)^{2}+s^{2}\left(x^{c}\right)d\Omega^{2},\ (c=0,1), \tag{1.10}$$

where u_A denotes the four-velocity of the fluid, and x^0 is the time-like coordinate. For a collapsing scalar field, on the othe hand, a possible choice of the gauge is

$$g_{01}\left(x^{c}\right)=0, \quad s\left(x^{c}\right)=x^{1}\equiv r, \tag{1.11}$$

so that the metric takes the form,

$$ds_{4}^{2}=g_{00}\left(t,r\right)dt^{2}+g_{11}\left(t,r\right)dr^{2}+r^{2}d\Omega^{2}, \tag{1.12}$$

where $t \equiv x^0$ and $\phi = \phi(t,r)$, with ϕ denoting the scalar field. Certainly, depending on a specific problem to be considered, other gauges can be chosen.

II. GAUGE FREEDOM AND GAUGE CHOICE FOR A SPHERICAL 3-BRANE

To begin with, consider the general action describing a 3-brane embedded in a D–dimensional bulk [12],

$$\begin{aligned} S=&\frac{1}{2\kappa_D^2}\int_{M_D} d^D x\sqrt{-{}^{(D)}g}\left({}^{(D)}R-2\Lambda_D+2\kappa_D^2\mathcal{L}_m^B\right)\\ &+\frac{1}{2\kappa^2}\int_{M_4} d^4x\sqrt{-g}\left(-2\Lambda+2\kappa^2\mathcal{L}_m\right), \end{aligned} \tag{2.1}$$

where ${}^{(D)}R$, Λ_D and $\mathcal{L}_m^B$ (R, Λ, $\mathcal{L}_m$) denote, respectively, the Ricci scalar, cosmological constant and matter content of the bulk (of the 3-brane). The constants κ_D and κ are related to the Planck scales M and M_{Pl}, respectively, by

$$\begin{aligned} \kappa_D^2 &= 8\pi\,{}^{(D)}G = M^{2-D},\\ \kappa^2 &= 8\pi G = M_{Pl}^{-2}, \end{aligned} \tag{2.2}$$

with $D \equiv n+4$. The bulk metric is ${}^{(D)}g_{MN}$ and $g_{\mu\nu}$ denotes the induced metric on the 3-brane, located on the surface $\Phi(x^A)=0$. Taking a different approach from SMS, we vary Eq. (2.1) with respect to ${}^{(D)}g_{MN}$ and $g_{\mu\nu}$ to get the full D–dimensional Einstein field equations in the form

$${}^{(D)}G_{MN}^{+}=\kappa_D^2 T_{MN}^{B+}-\Lambda_D\,{}^{(D)}g_{MN}^{+},\ (\Phi\geq 0), \tag{2.3}$$

$${}^{(D)}G_{MN}^{-}=\kappa_D^2 T_{MN}^{B-}-\Lambda_D\,{}^{(D)}g_{MN}^{-},\ (\Phi\leq 0), \tag{2.4}$$

$${}^{(4)}G_{\mu\nu}^{Im}=\kappa_D^2\left(T_{\mu\nu}-\frac{\Lambda}{\kappa^2}g_{\mu\nu}\right),\ (\Phi=0), \tag{2.5}$$

where T_{MN}^{B} and $T_{\mu\nu}$ denote the stress energy tensor of the bulk and of the 3-brane, respectively. The quantities with superscript "+" ("−") denote those calculated in the region $\Phi \geq 0$ ($\Phi \leq 0$), and ${}^{(4)}G_{\mu\nu}^{Im}$ denotes the delta-function-like (impulsive) part of G_{MN} with support on the 3-brane,

$${}^{(D)}G_{MN}={}^{(D)}G_{MN}^{+}H\left(\Phi\right)+{}^{(D)}G_{MN}^{-}\left[1-H\left(\Phi\right)\right]+{}^{(4)}G_{\mu\nu}^{Im}\delta_M^{\mu}\delta_N^{\nu}\delta\left(\Phi\right), \tag{2.6}$$

where $\delta\left(\Phi\right)$ denotes the Dirac delta function, and $H(\Phi)$ the Heavside function, defined as

$$H\left(\Phi\right)=\begin{cases}1, & \Phi\geq 0,\\ 0, & \Phi<0.\end{cases} \tag{2.7}$$

Eq.(2.6) can easily be obtained by the following considerations. Let us first denote the region with $\Phi \geq 0$ as V^+, the region with $\Phi \leq 0$ as V^-, and the hypersurface $\Phi = 0$ as Σ. Then, since the Einstein field equations involve the second-order derivatives of the metric coefficients, one can see that the metric must be at least C^2 in regions $V^{\pm}$ and C^0 across the hypersurface $\Phi = 0$, so that the Einstein field equations (or any second-order differential equations

involved) hold in the sense of distributions [13]. Consequently, the metric g_{AB} in the whole spacetime can be written as

$$g_{AB} = g^{+}_{AB} H\left(\Phi\right) + g^{-}_{AB}\left[1 - H\left(\Phi\right)\right], \tag{2.8}$$

where quantities with superscripts " + " (" − ") denote the ones defined in V^{+} (V^{-}). Hence, we find that

$$\begin{aligned} g_{AB,C} &= g^{+}_{AB,C} H\left(\Phi\right) + g^{-}_{AB,C}\left[1 - H\left(\Phi\right)\right], \\ g_{AB,CD} &= g^{+}_{AB,CD} H\left(\Phi\right) + g^{-}_{AB,CD}\left[1 - H\left(\Phi\right)\right] + \left[g_{AB,C}\right]^{-} \Phi_{,D}\delta\left(\Phi\right), \end{aligned} \tag{2.9}$$

where $(\ )_{,C} \equiv \partial(\ )/\partial x^{C}$ and

$$\left[g_{AB,C}\right]^{-} \equiv \lim_{\Phi\to 0^{+}} \frac{\partial g^{+}_{AB}}{\partial x^{C}} - \lim_{\Phi\to 0^{-}} \frac{\partial g^{-}_{AB}}{\partial x^{C}}. \tag{2.10}$$

From Eq.(2.9) and the following,

$$\begin{aligned} & H^{m}\left(\Phi\right) = H\left(\Phi\right), \quad \left[1 - H\left(\Phi\right)\right]^{m} = 1 - H\left(\Phi\right), \\ & H\left(\Phi\right)\left[1 - H\left(\Phi\right)\right] = 0, \quad \left[1 - H\left(\Phi\right)\right]\delta(\Phi) = \frac{1}{2}\delta(\Phi) = H\left(\Phi\right)\delta(\Phi), \end{aligned} \tag{2.11}$$

where m is an integer, we can easily see that the Einstein tensor G_{AB} can be written, in general, in the form of Eq.(2.6).

In the 5D case a 3-brane is a hypersurface of a 5D bulk, and one can show that Eq. (2.5) is identical to Eq. (1.4) when written out in terms of the extrinsic curvature of the 3-brane [13]. Similarly, using the Gauss–Codacci relations one can show that Eqs. (1.1) and (1.2) follow from Eqs. (2.3) and (2.4). It is, at the same time, worth emphasizing that Eqs. (2.3–2.4) contain additional information not present in Eqs. (1.1–1.2). This includes, for instance, information about the evolution of $\mathcal{E}_{\mathcal{AB}}$. For the case that $D \geq 6$, a 3-brane is a surface of co–dimension $(D-4)$ with respect to the bulk, and the problem becomes more subtle. In particular, the generalization of the Gauss–Codacci relations and the Israel junction conditions to these cases has not, to our knowledge, been worked out. For this reason, in the $D \geq 6$ cases we will only consider models with additional symmetries. In this way, a 3-brane can be considered as a degenerate hypersurface. Indeed, this assumption turns out to include most of the braneworld models with higher dimensional bulks which have been studied so far [3].

A. $D = 5$

Considering first the case $D = 5$, the most general bulk metric with a S^2 symmetry takes the form,

$$ds^2 = g_{ij}\left(x^k\right) dx^i dx^j + s^2\left(x^k\right) d\Omega^2, \tag{2.12}$$

where $i,\ j$ and k are taken here to range over $0,\ 1$ and 2 with $z \equiv x^2$. The location of a 3-brane with spherical symmetry in general can be written as

$$\Phi\left(x^0, x^1, z\right) = 0. \tag{2.13}$$

Note that the form of the metric (2.12) is invariant under the coordinate transformations,

$$x^i = f^i\left(\bar{x}^j\right),\ (i, j = 0, 1, 2). \tag{2.14}$$

As a result, using these three degrees of freedom, we can choose coordinates such that the brane is always located on the hypersurface $z = 0$ and

$$\Phi\left(x^0, x^1, z\right) = z, \quad g_{0z}(x^0, x^1, z) = g_{1z}(x^0, x^1, z) = 0. \tag{2.15}$$

This choice of coordinates will be referred to as the *canonical gauge*.

Using this form for the metric together with the definition of the extrinsic curvature, $K_{AB} \equiv h_A{}^C h_B{}^D \nabla_C n_D$, we find that

$$K_{\mu\nu} = (2N)^{-1}\frac{\partial g_{\mu\nu}(x^{\alpha}, z)}{\partial z}, \tag{2.16}$$

where $n_A = N\delta^z_A,\ N \equiv \sqrt{g_{zz}}$, and ∇_C denotes the covariant derivative with respect to ${}^{(D)}g_{AB}$. For the metric (2.12) in the canonical gauge we find that the Israel junction conditions, (1.4), yield

$$T_{\mu\nu} = \lambda g_{\mu\nu} - \frac{1}{2\kappa_5^2 N}\left([g_{\mu\nu,z}]^- - g_{\mu\nu}g^{\alpha\beta}[g_{\alpha\beta,z}]^-\right). \tag{2.17}$$

It should be noted that even in the canonical gauge, there is residual coordinate freedom on the 3-brane:

$$x^0 = F^0(\bar{x}^0, \bar{x}^1), \quad x^1 = F^1(\bar{x}^0, \bar{x}^1). \tag{2.18}$$

We can exploit this remaining freedom and set

$$g_{01}(x^0, x^1, 0) = g_{01,z}(x^0, x^1, 0) = 0, \tag{2.19}$$

so that the reduced metric on the 3-brane takes the form,

$$ds^2\big|_{z=0} = \gamma_{00}(x^0, x^1)\left(dx^0\right)^2 + \gamma_{11}(x^0, x^1)\left(dx^1\right)^2 + \gamma_{22}(x^0, x^1)d\Omega^2, \tag{2.20}$$

where $\gamma_{ab}(x^0, x^1) \equiv g_{ab}(x^0, x^1, 0)$. It is remarkable to note that *Eq.(2.19) is possible only in the cases where one of the three energy conditions, weak, strong and dominant, holds* [17]. To show this, following Chandrasekhar [14], we first transform to the coordinates $\bar{t}$ and $\bar{r}$ using the transformations $x^0 = \phi(\bar{t}, \bar{r})$ and $x^1 = \psi(\bar{t}, \bar{r})$, in which the metric takes the form

$$ds^2 = -b\, d\bar{t}^2 + 2c\, d\bar{t}d\bar{r} + d\, d\bar{r}^2 + s^2 d\Omega^2 + N^2 dz^2, \tag{2.21}$$

where $b,\ c,\ d$, and s are functions of $\bar{t},\ \bar{r}$ and z, and have the properties [14],

$$b(\bar{t}, \bar{r}, 0) = d(\bar{t}, \bar{r}, 0), \quad c(\bar{t}, \bar{r}, 0) = 0, \tag{2.22}$$

at $z = 0$. If we now make another coordinate transformation $\bar{t} = \Phi(t, r),\ \bar{r} = \Psi(t, r)$, it is straightforward to show that the conditions $g_{tr}(t, r, 0) = 0$ and $g_{tr,z}(t, r, 0) = 0$ reduce to

$$\Phi_{,t} = A\,\Psi_{,t}, \tag{2.23}$$
$$\Phi_{,r} = A^{-1}\,\Psi_{,r}, \tag{2.24}$$
$$(A\Psi_{,t})_{,r} = (A^{-1}\,\Psi_{,r})_{,t}, \tag{2.25}$$

where

$$\begin{aligned} A(t, r) &\equiv \frac{1}{2c_{,z}}\left[(b_{,z} - d_{,z}) \pm \Delta^{1/2}\right]_{z=0}, \\ \Delta(t, r) &\equiv (b_{,z} - d_{,z})^2 - 4{c_{,z}}^2\,|_{z=0}. \end{aligned} \tag{2.26}$$

Eq.(2.25) represents the integrability condition of Eqs.(2.23) and (2.24). Since the metric coefficients is at least C^2 in regions $V^{\pm}$ and C^0 across the hypersurface $z = 0$, we can see that, with respect to t and r, the metric is also at least C^2 even across the hypersurface $z = 0$. For such a C^2 metric, the theorems given in [15] show that there will always exist a region of the (t, r)-plane for which Eqs.(2.23)-(2.25) have solutions. However, such solutions will be real only if

$$\Delta \geq 0. \tag{2.27}$$

This condition is ensured by assuming any of the standard energy conditions [17]. Indeed, using the Israel junction conditions (1.4), we find

$$\Delta = (\kappa_5 bN)^2\left[(\rho + p_{\bar{r}})^2 - 4q^2\right], \tag{2.28}$$

where

$$\rho \equiv \frac{T_{\bar{t}\,\bar{t}}}{b}, \quad p_{\bar{r}} \equiv \frac{T_{\bar{r}\,\bar{r}}}{d}, \quad q \equiv \frac{T_{\bar{t}\,\bar{r}}}{(bd)^{1/2}}. \tag{2.29}$$

As shown in [16], *a necessary condition for any of the weak, dominant, and strong energy conditions to hold is* $\Delta \geq 0$. We thus conclude that *a coordinate transformation exists such that (2.19) holds, provided that one of the three energy conditions holds.*

As a consequence of our coordinates and Eq. (2.17), $T_{\mu\nu}$ must be diagonal. In particular, we have

$$T_{tr}(t,r) = 0. \tag{2.30}$$

This represents a very strong restriction on the dependence of matter fields confined to the 3-brane on the spacetime coordinates. To see this clearly, let us first consider a scalar field ϕ, for which the stress tensor is given by

$$T^{\phi}_{\mu\nu} = \nabla_\mu\phi\nabla_\nu\phi - \frac{1}{2}g_{\mu\nu}[(\nabla\phi)^2 + 2V(\phi)]. \tag{2.31}$$

In this case, we have

$$\Delta = b^{-2}({\phi_t}^2 - {\phi_r}^2)^2, \quad T^{\phi}_{tr}(t,r) = \phi_{,t}(t,r)\phi_{,r}(t,r). \tag{2.32}$$

Then, Eq.(2.30) implies

$$(i)\ \ \phi(t,r) = \phi(t), \quad \text{or} \quad (ii)\ \ \phi(t,r) = \phi(r). \tag{2.33}$$

One can show that this is also true for a spherically symmetric $SU(2)$ Yang-Mills field. In that case the relevant term is

$$T^{\mathrm{YM}}_{tr} \propto w_{,t}(t,r)w_{,r}(t,r), \tag{2.34}$$

where w is the Yang-Mills potential [18].

Similarly, if one considers a perfect fluid with stress tensor

$$T^{\mathrm{fl}}_{\mu\nu} = (\rho + p)u_\mu u_\nu + p\, g_{\mu\nu}, \tag{2.35}$$

for which

$$\Delta = (\rho + p)^2, \quad T^{\mathrm{fl}}_{tr} = (\rho + p)u_t u_r, \tag{2.36}$$

it turns out that the fluid cannot have a radial velocity, that is, for the present choice of the gauge, we must have

$$u_r = 0. \tag{2.37}$$

It is interesting to note that the condition (2.27) is satisfied for the cosmological constant, for which the corresponding energy-momentum tensor is given by Eq.(2.35) with $p = -\rho = \lambda$.

The above results hold not only for $D = 5$ but also for $D \geq 6$. To see this, in the following let us first consider the case where the 3-brane is a surface of co–dimension two, that is, $D = 6$. Then, we shall further generalize our results to the case $D > 6$.

B. $D = 6$

Because we now have two extra spatial directions, the generalization of the RS model to include matter fields on the 3-brane becomes non-trivial. Following Israel [19], we will assume a cylindrical symmetry in these extra dimensions and that the 3-brane is located on the 4-dimensional surface $\rho = 0$ where ρ and ψ are chosen as polar-like coordinates for the two extra dimensions, and $\rho = 0$ is the symmetry axis for the cylindrical symmetry. This includes most of the braneworld models in six dimensional spacetimes [3].

Under these assumptions, it can be shown that the general bulk metric with a spherical 3-brane can be cast in the form

$$\begin{aligned} ds^2 = &-\alpha^2 dt^2 + a^2(dr + \beta dt)^2 + s^2 d\Omega^2 \\ &+ N^2 d\rho^2 + f^2\rho^2\left(d\psi + \omega dt\right)^2 + g^2\rho^2 d\psi^2, \end{aligned} \tag{2.38}$$

where ω represents the rotation of the 3-brane, and all the metric coefficients are functions of $t,\ r$ and ρ, subject to the gauge (2.19). Because the symmetry axis now represents a 3-brane, certain conditions must be imposed there [20]. For the present purpose, it is sufficient to assume that: (a) the symmetry axis must exist, that is,

$$X \equiv ||\partial_\psi|| \to O(\rho^2), \tag{2.39}$$

as $\rho \to 0$, where ∂_ψ is the cylindrical Killing vector with closed orbits, and that (b) the spacetime is free of curvature singularities on the axis, which can be assured by assuming the local flatness condition,

$$\lim_{\rho \to 0^+} \frac{X_{,A} X_{,B} \, {}^{(D)}g^{AB}}{4X} = 1. \tag{2.40}$$

To generalize Israel's method [5] to this case, we first calculate the extrinsic curvature K_{ab} of the hypersurface $\rho = \epsilon$ and then take the limit $\epsilon \to 0$. Introducing the quantities $\mathcal{K}_{ab}$ by

$$\mathcal{K}_{ab} \equiv \lim_{\rho \to 0^+} \left(\sqrt{-{}^{(5)}g} \, K_{ab} \right), \tag{2.41}$$

the surface stress energy tensor can be defined as [19]

$$T^a_b = -\left(\mathcal{K}^a_b - \delta^a_b \mathcal{K}^c_c \right), \tag{2.42}$$

provided $\mathcal{K}^a_{[c} \mathcal{K}^b_{d]} \neq 0$ and where we have let $a, \ b = 0, \cdots, 4$ and $x^5 = \rho$. In this case, it can be shown that the extrinsic curvature of the hypersurface $\rho = \epsilon$ is given by

$$K_{ab} = \frac{1}{2N} \frac{\partial g_{ab}(x^c, \epsilon)}{\partial \rho}. \tag{2.43}$$

For the case $\mathcal{K}^a_{[c} \mathcal{K}^b_{d]} = 0$, the surface stress energy tensor is instead defined as [19],

$$T^a_b = \frac{2\pi}{N^2 \sqrt{f^2 + g^2}} \left(\sqrt{f^2 + g^2} - N \right) \delta^a_\mu \delta^\nu_b, \tag{2.44}$$

where $\mu, \nu = 0, \cdots 3$. In passing, we note this case also corresponds to a cosmic string in a 6D bulk [21].

However, in each of the above two cases it can be seen that the condition Eq.(2.30) holds. Therefore, the junction conditions across the 3-brane in a 6D bulk yield the same restrictions on the dependence of the spacetime coordinates of the matter fields confined on the 3-brane as those in the 5D case.

C. $D > 6$

When $D > 6$, we consider only the case where the extra n–dimensional space has an $SO(n-1)$ symmetry so that the bulk metric can be written in the form,

$$\begin{aligned} ds^2 = & -\alpha^2 dt^2 + a^2 (dr + \beta dt)^2 + s^2 d\Omega^2 \\ & + N^2 \left(d\rho^2 + \rho^2 d\Omega^2_{n-1} \right), \end{aligned} \tag{2.45}$$

where $d\Omega^2_{n-1}$ denotes the metric of a unit $(n-1)$–dimensional sphere, and all the metric coefficients are functions of t, r and ρ. Using, as before, the coordinate freedom $t = F_1(t', r')$ and $r = F_2(t', r')$ we can always assume that Eq. (2.19) holds. From Eq. (2.45) we note that $\rho = 0$ represents a four-dimensional spacetime with spherical symmetry. This we shall take as our 3-brane. Certainly, this is acceptable only after some (physical and or geometrical) conditions are satisfied at $\rho = 0$. These will include, as before, that the spacetime is free of curvature singularities there. In order to generalize Israel's method to this case, we also require that the limit

$$\mathcal{K}_{ab} = \lim_{\rho \to 0^+} \left(\sqrt{-{}^{(D-1)}g} \, K_{ab} \right), \tag{2.46}$$

exists, where K_{ab} denotes the extrinsic curvature of the hypersurface $\rho = \epsilon$, but now with $a, b = 0, 1, \cdots, D-2$, and $x^{D-1} = \rho$. With these conditions, we can define the surface stress energy tensor as that given by Eqs. (2.42) and (2.44). For the metric (2.45), it can be shown that the extrinsic curvature K_{ab} of the hypersurface $\rho = \epsilon$ is also given by Eq. (2.43). Substituting it into Eq. (2.42), we find again that the component $T_{tr}(t, r)$ vanishes identically for both of the cases described by Eqs. (2.42) and (2.44). Thus, the same restrictions on the dependence of the spacetime coordinates of the matter fields confined to the 3-brane that occur in the 5D case continue to hold for a D–dimensional bulk given by metric (2.45).

III. CONCLUSIONS

In summary, we have studied the embedding of a spherically symmetric 3-brane into a D–dimensional bulk with arbitrary matter fields both on the brane and in the bulk in the context of the braneworld scenario. We have found that, for a particular choice of gauge, the boundary (Israel's junction) conditions across the brane together with imposition of the weak energy condition provide very strong restrictions on the dependence of matter fields confined to the 3-brane on the spacetime coordinates. As examples, a scalar field or a Yang-Mills field can be only either time-dependent or radial-coordinate dependent, while for a perfect fluid its radial velocity must vanish.

In this paper, we have studied only the function dependence of the metric coefficients for the canonical gauge. It would be very interesting to study the effects of the matter fields in the bulk on the effective 4-dimensional energy-momentum tensor ${}^{(4)}T_{AB}$, a subject that is under our current investigation. Another interesting problem is the applications of the results obtained in this paper to cosmology [22].

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